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Nonregular languages in the kicked rotor

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A symbolic dynamics of the Markov chain model for trajectories of the kicked rotor (standard map) in sticky regions near resonant orbits is developed. Two sets of trajectories are shown to correspond to a context-free and a context-sensitive language, respectively.

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The connection between computation and the time evolution of dynamical systems has been under close scrutiny by physicists for well over a decade ([1–4] and references therein). In particular, there have been several studies relating symbolic dynamics, a particular encoding of time-series data from experimental and mathematical models, to the Chomsky language hierarchy [4,5], which consists of four levels of languages, defined by the amount of computational power (time or memory) needed to recognize that a particular string is an instance of the language in question. The grammar classes are regular, context-free, context-sensitive, and unrestricted. Several experimental studies [6] have shown symbolic dynamics of real-life (e.g., chemical, laser, fluids) systems that are described by regular languages both in the ordered and in the chaotic regimes (see also [7]), while the transition from order to chaos in the same systems is described by Morse-Thue sequences, which have a computational requirement broadly in the context-free class [4,8].

In this Rapid Communication we show that two sets of trajectories of the standard map, a physically realizable Hamiltonian dynamical system (see [9] for a detailed introduction), can be described by grammars that are above regular. We do this by developing a different three-symbol dynamics for trajectories in the sticky region near a resonant orbit, where a (multi)fractal structure of islands exists. As in dissipative systems, we find the higher-level languages in the parameter region which corresponds to the order-chaos transition (the transition to global stochasticity, in this case). However, in Hamiltonian systems such as the standard map, periodic orbits and the globally stochastic region (both presumably described by a regular language) coexist with the stickiness region that concerns us. Therefore, in our example the complex behavior corresponds to a restricted set of initial conditions. A similar role for initial conditions in the classification of complexity classes has been previously reported in cellular automata models [10]. As we will show, our symbolic dynamics is closely related to the calculation of transport properties of the standard map (see [11,12]).

Before presenting our languages, we will review some aspects of both computational languages and the standard map. The Chomsky hierarchy can be explained in terms of the memory requirements of a finite-state automaton designed to generate (or recognize) instances of the particular language. Regular languages correspond to paths along a directed graph that has no special memory requirements. A word in the language is generated by just adding symbols at the end of a shorter, allowed word [7]. Context-free languages can be generated by substitution rules, and are therefore parallel rather than sequential in nature (which makes it intriguing that iterated maps can generate them). They require memory in the form of a stack. Context-sensitive languages also correspond to substitutions, but with rules which are sensitive on positions in the string near to that where the substitution is made (hence the name). These can be generated or recognized by automata with a memory size proportional to the length of the string to be recognized (linear bound automata). Finally, unrestricted languages are generated by Turing machines, or automata with an infinite amount of memory tape. These are able to generate all computable functions. We remark that the Chomsky hierarchy is a broad characterization of the computational power of languages and grammars; there are more detailed studies of specific computational requirements such as number of stacks or queues or different functional dependences of memory tape on input string length: see [13] for one-dimensional iterated maps.

The standard map,

$$I_{n+1}=I_n+K\sin\theta_n, \tag{1}$$

$$\theta_{n+1}=\theta_n+I_{n+1}, \tag{2}$$

is perhaps the best-known Hamiltonian chaotic map. With I denoting angular momentum, θ its conjugate angle, and K the strength of the kick (stochasticity parameter), this map represents a Poincaré section of a periodically kicked rotor. The standard map also approximates other physical situations, such as the Fermi accelerator model [9].

In the present work we are interested in K values around the transition to global stochasticity, for which the phase space is a complicated mixture of invariant [Kolmogorov-Arnol'd-Moser (KAM)] curves, broken or not, islands (librational curves), and chaotic regions. As K is increased from zero, the rotational circles (invariant curves that encircle the phase space cylinder) with rational rotation frequency break and become island chains; those with irrational frequency last longer, but eventually become cantori (invariant Cantor sets), with the most irrational, as defined by the Diophantine condition, being destroyed last. Through this process, the regions between islands become chaotic, and gradually con-

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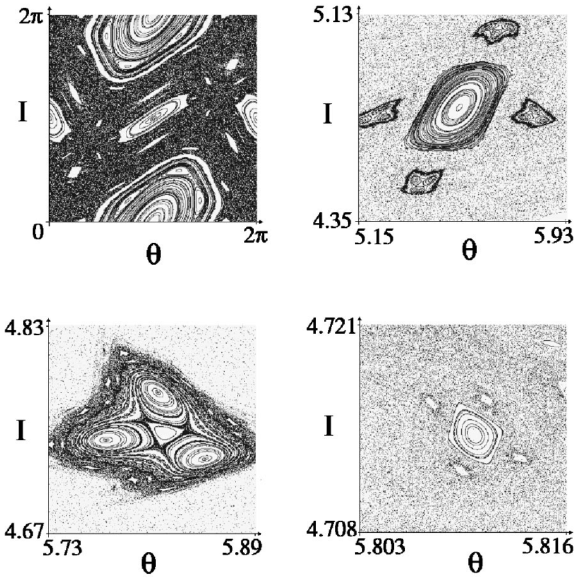

FIG. 1. Island structure for the standard map with $K=1.3$. The figures are successive zooms of phase space, and should be read in the same order as ordinary text.

nect until the entire phase space is chaotic. Figure 1 illustrates the nearly self-similar phase space in this regime through successive zooms of the islands at $K=1.3$.

It has been observed that islands are sticky: since the motion has to be locally continuous, trajectories on the chaotic sea that are close to KAM curves must stay close for some time. In this regime, surprisingly, even on a connected chaotic component the distribution is not uniform for arbitrarily long times. This leads to anomalous diffusive behavior due to the nonvanishing contributions of the sticky zones, which so far cannot be handled even by the most sophisticated calculations [11]. It is in the search for a full understanding of this phenomenon that the Markov chain and tree models appear, and with them the connection to the new symbolic dynamics. While symbolic dynamics have been developed for the standard map both in the regular regime and the stochastic regime by constructing partitions, for example, from homoclinic tangencies and fibers of invariant manifolds (see [14] and references therein), here we concentrate on the symbolic dynamics for the sticky region. Our approach is a modification of the language developed by Meiss [15], described below. The main difference is that we use strings to represent trajectories and not state labels.

Consider a trajectory that starts on the chaotic component and gets close to an island. When invariant circles break up to form cantori, the area of phase space that can cross a broken invariant circle in one iteration (flux) increases gradually. It has been observed that diffusion occurs in jumps [16] across cantori; one can choose which minimum level of flux defines the cantori that are considered. Within each region surrounded by these cantori the motion is approximately random, so transport can be modeled as a stochastic process where the states coincide with the forementioned regions; the states can be counted starting in an invariant circle, an island or any given region. Transitions in or out of a region can occur only through an area defined by the stable and unstable manifolds of the end points of a cantorus, known as the turnstile.

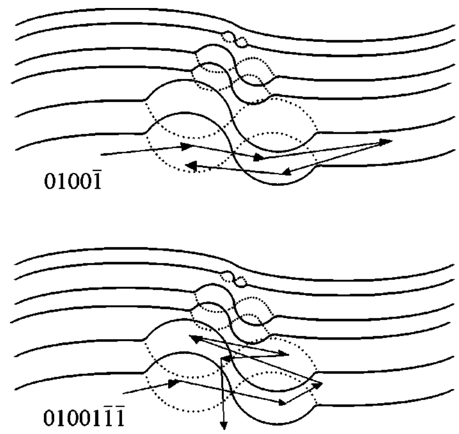

FIG. 2. Abstract representation of I-θ space in the sticky region. The symbolic dynamics for two exiting trajectories in the Markov chain model are shown. The ∞-shaped regions where transitions can occur are the turnstiles.

Two models can be constructed on this basis. In the Markov chain model, the probability of getting closer to the island is less than the probability of getting farther, so the long time it takes a trajectory to leave a deep level is compensated directly by the small probability of getting deep inside the structure. In the Markov tree model, there is also the possibility of going inside (and eventually leaving) a substructure, subsubstructure, etc. This is consistent with the long time trapping that has been observed, and has important consequences: it predicts a powerlike decay of correlations [15], and hence the nonexistence of a diffusion coefficient.

Now, consider a trajectory in the sticky region near a particular cantorus. The quantity of interest is the distance to the cantorus in the structure. Fix a region (where the trajectory starts) as the origin. The language associated with this model (Markov chain) consists of three symbols, 1, 0, and $\bar{1}$. 0 means the trajectory stayed in the same region as in the previous time step and 1 ($\bar{1}$) means it has moved one level closer to (farther from) the cantorus. This is illustrated in Fig. 2. In contrast, the respective Meiss symbolic dynamics would label the location at individual time steps with one 1 for each level that has gone in. In the Markov tree model there are two additional symbols, 2 ($\bar{2}$), to indicate that the trajectory has gone inside (outside) the nearest island chain. Note that since the minimum-flux cantori are to a certain degree arbitrarily chosen (and particularly the outer one), so is the sticky region itself.

We define p_{nm} to be the probability that a trajectory has advanced exactly to the nth island and exited the sticky region after exactly m time steps. For a given string of length m, recognizing if it is an exiting trajectory is a context-free procedure (Ref. [17], p. 13) that can be performed by a one-counter automaton, or alternatively, by a stochastic pushdown automaton [18]. Each p_{nm} can be calculated by summing probabilities over all individual trajectories of length m that satisfy the condition on n. Knowing the p_{nm} one can calculate the probability that a trajectory is trapped during exactly m time steps,

$$p(m)=\sum_{n=2}^{[m/2]} p_{nm}, \tag{3}$$

as well as the probability that a trajectory advances to exactly the nth level and exits,

$$p(n)=\sum_{m=2n}^{\infty} p_{nm}. \tag{4}$$

To calculate the transition probabilities between two regions i and j, two quantities are needed: the area of phase space of the i state and area of the overlap between the turnstiles. For simplicity, only nearest level transitions will be taken into account, so only the area of each turnstile is required. The calculation of the transition probabilities in general has to be calculated numerically. In the case of the golden mean cantorus, the structure is simply self-similar, and the flux (ΔW_i) and area (A_i) are

$$\Delta W_i=\Delta W_0 a^i, \tag{5}$$

$$A_i=A_0 b^i, \tag{6}$$

where ΔW_0 and A_0 depend on the reference state (the beginning of the string) and the a and b are obtained through renormalization group theory [16].

A possible truncation in the calculation of the p_{nm}, motivated by the fact that the probabilities of going deeper into the structure are smaller, is to only include exiting trajectories for which the first maximum position is n. The recognition of symbolic strings that satisfy this requirement is closely related to the example given in [4], pp. 175–177: the language of 1D random walks that are confined in the region between the origin (O) and the first maximum position ($u_{\max}$) and that come back to O. This language can be simulated with a set of 13 production rules and 13 symbols, which serve the role of keeping track of the maximum position (R), of the current position of the random walker (U), and the difference between both. Three of the 13 production rules involve two terminal symbols on the left side, which makes the language context-sensitive. Alternatively, an equivalent two-counter finite-state automaton can be constructed to recognize the language.

To transform this to the Markov chain model we need an additional symbol in the language, 0, which allows the position counters (R, U) to pass through in either the left or right direction without altering the current state of the finite automaton or the memory tape. This shows that deciding which strings to include under the truncation just introduced is a context-sensitive process. Also, by going from a random walk to the Markov chain model, *we are now considering a dynamical system*.

We have not been able to perform a full analysis of the Markov tree model's language. First of all, the language is not in the unrestricted class, since even with levels and islands it is clearly decidable whether or not a finite-length trajectory satisfies the maximum position and exiting conditions. In order to keep track of walks in the treelike structure of cantori, one would need k stack symbols, where k is the degree of the tree. In the unrestricted language we would still have a context-free language, but the language restricted to the first maximum position would be context-senstive.

The identification and counting of the strings and their probabilities, as used in Eq. (3), can be used to calculate the (average) trapping time for a given region if the trajectory starts inside the structure, and the fluctuations in density if the trajectory starts far from the island or curve. A proper calculation would require the identification and summation of *all* the p_{nm} terms in this equation, or the truncation which we have proposed.

If a globally chaotic component exists, this approach allows the calculation of the transport coefficients; if a diffusion coefficient is not adequate, the quantity of interest is the exponent of the (powerlike) time dependence of the rms distance to the original position. This has been numerically determined [19] to be around $t^{1.4}$ and is approximated by these models: the chain model gives twice the value of the exponent, while the tree model gives the correct answer. The result can be calculated by summing over the strings that join two given states, weighted by the length of each string.

The results of this Rapid Communication can be summarized as follows: we have developed a symbolic dynamics for trajectories in the Markov tree and Markov chain models of motion in the sticky region near islands of the standard map, a physically realizable dynamical system. These trajectories are relevant for the calculation of transport properties in the transition regime between order and chaos. We have defined languages for the orbits which exit the sticky region in the Markov chain approximation. Depending on whether all trajectories are considered, or just those in a restricted set, we have respectively a context-free and a context-sensitive language.

Our finding of high-level languages in the sticky region is consistent both with the complexity of the behavior of the trajectories and with the difficulty to perform calculations of the transport properties of the system. We remark that the chain and tree models have a metric element as well as a topological one, since the transition probabilities (equivalently, the occurrence of nonzero symbols) depend on which level is occupied by the trajectory at the time.

The examples in question share two features previously observed in dynamical systems: (1) the symbolic dynamics only applies to a particular set of initial conditions, as was also observed in Ref. [10], and (2) the high-complexity languages occur near the order-chaos transition, as has been reported for the period-doubling route to chaos [8].

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